Limitations of Expressive Power of First-Order Logic

In this lecture we assume that there are no function symbols in the signature

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Informally: QR is the nesting depth of quantifiers in the formula

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- for $r \in \Sigma_n^R$ we define $r^{\mathfrak{A}_{|B}} := r^{\mathfrak{A}} \cap B^n$.

Partial isomorphism

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Its domain is $dom(h) = A'$, and range is $rg(h) = B'$.

Partial isomorphisms cntd

We adopt the convention that $\emptyset$ is a partial isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ with empty domain and range.

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For two partial isomorphisms g, h from $\mathfrak{A}$ to $\mathfrak{B}$ we write $g \subseteq h$ when $dom(g) \subseteq dom(h)$ and $g(a) = h(a)$ for all $a \in dom(g)$; alternatively, when g is included in h as a set.

m-isomorphism

Let $m \in \mathbb{N}$.

Structures $\mathfrak{A}$ and $\mathfrak{B}$ are m-isomorphic (dentoed $\mathfrak{A} \cong_m \mathfrak{B}$), if there exists a family $\{I_n \mid n \leq m\}$ such that:

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- **Forth** For each $h \in I_{n+1}$ and each $a \in A$ there esxists $g \in I_n$ such that $h \subseteq g$ and $a \in dom(g)$.

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The family $\{I_n \mid n \leq m\}$ is called an <u>m-isomorphism</u> of $\mathfrak{A}$ and $\mathfrak{B}$, denoted $\{I_n \mid n \leq m\} : \mathfrak{A} \cong_m \mathfrak{B}$.

Finite isomorphism

Two structures $\mathfrak{A}$, $\mathfrak{B}$ are finitely isomorphic, (denoted $\mathfrak{A} \cong_{fin} \mathfrak{B}$) if there exists a family $\{I_n \mid n \in \mathbb{N}\}$, whose each subfamily $\{I_n \mid n \leq m\}$ is an m-isomorphism.

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If $\{I_n \mid n \leq m\}$ has the above property, we write
$\{I_n \mid n \leq \mathbb{N}\} : \mathfrak{A} \cong_{fin} \mathfrak{B}$
This family is called a finite isomorphism.

Finite isomorphism

Fakt

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- If $\mathfrak{A} \cong \mathfrak{B}$, then $\mathfrak{A} \cong_{fin} \mathfrak{B}$.
- If $\mathfrak{A} \cong_{fin} \mathfrak{B}$ and the universe A of $\mathfrak{A}$ is finite, then $\mathfrak{A} \cong \mathfrak{B}$.

Elementary equivalence

Repetitio est mater studiorum

$\mathfrak{A}$ and $\mathfrak{B}$ are elementary equivalent (denoted $\mathfrak{A} \equiv \mathfrak{B}$), if for each sentence φ of first-order logic

$\mathfrak{A} \models \varphi$ iff $\mathfrak{B} \models \varphi$.

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$\mathfrak{A}$ and $\mathfrak{B}$ are m-elementary equivalent (denoted $\mathfrak{A} \equiv_m \mathfrak{B}$), if for each sentence φ of quantifier rank at most m holds
$\mathfrak{A} \models \varphi$ iff $\mathfrak{B} \models \varphi$.

Elementary equivalence cntd

Fact

$\mathfrak{A} \cong_{fin} \mathfrak{B}$ if and only if for every natural m holds $\mathfrak{A} \cong_m \mathfrak{B}$.

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$\mathfrak{A} \cong_{fin} \mathfrak{B}$ if and only if for every natural m holds $\mathfrak{A} \cong_m \mathfrak{B}$.

Proof:

Suppose that for each m there exists $\{I_n^m \mid n \leq m\}$ as in the definition of $\cong_m$.

The family $\{J_n \mid n \in \mathbb{N}\}$ defined by

$$J_n = \bigcup_{m \in \mathbb{N}} I_n^m$$

satisfies the definition of $\cong_{fin}$.

Fraïssé's characterization

Theorem [Fraïssé]
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  - $QR(\varphi) \leq n \leq m$

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The for each $a_1, \ldots, a_r \in dom(g)$ the following are equivalent:

$$\mathfrak{A}, x_1 : a_1, \ldots, x_r : a_r \models \varphi$$

$$\mathfrak{B}, x_1 : g(a_1), \ldots, x_r : g(a_r) \models \varphi.$$

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  - $(\mathfrak{B}, x_1 : g(a_1), \ldots, x_r : g(a_r)) \models \varphi$.

An example of application

Fakt

If $\mathfrak{A}, \mathfrak{B}$ are two finite linear orders of cardinalities $> 2^m$, then $\mathfrak{A} \equiv_m \mathfrak{B}$.

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For $k \leq m$ we define "distance" d_k between elements by

$$d_k(a, b) = \begin{cases} |b - a| & \text{if } |b - a| < 2^k \\ \infty & \text{otherwise} \end{cases}$$

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Now see blackboard.

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- Player II chooses

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- Player I chooses:
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  - an element of its universe (denoted a_i if from A, b_i if from B)
- Player II chooses
  - teh other structure
  - an element of its universe (denoted a_i if from A, b_i if from B)

And the winner is. . .

And the winner is...

In the m rounds the chosen elements are $a_1, \ldots, a_m \in A$ and $b_1, \ldots, b_m \in B$.

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Player II has a winning strategy in $G_m(\mathfrak{A}, \mathfrak{B})$, if he/she can win any play, irrespectively of the moves of Player I.

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Game application example (easy)

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The following graphs can be distinguished by a sentence of quantifie rrank 4, but rank 3 is not sufficient.

```
  *      *      *      *           *      *      *
  |      |      |      |           |      |      |
*-*-*  *-*-*  *-*-*  *-*-*       *-*-*  *-*-*  *-*-*
  |      |      |      |           |      |      |
  *      *      *      *           *      *      *
```

Game application example (harder)

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Proof: See blackboard

Corollary

$\langle \mathbb{R}, \leq \rangle \equiv \langle \mathbb{Q}, \leq \rangle.$

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There is no sentence of first-order logic which distinguishes continuous linear orders from noncontinuous ones

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- $\mathbf{Th}(\mathfrak{A}) = \{\varphi \mid \mathfrak{A} \models \varphi\}$ (theory of a model $\mathfrak{A}$).

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Corollary (of the last theorem) Theore of the class $\mathcal{A}$ of all dense linear orders without maximum and minimum is complete